J. Fluid Mech. (1981), *vol.* 106, *pp.* 131–147
Printed in Great Britain

The Korteweg–de Vries equation: a historical essay

By JOHN W. MILES

Institute of Geophysics and Planetary Physics,
University of California, La Jolla, California 92093

The Korteweg–de Vries (KdV) equation, usually attributed to Korteweg & de Vries (1895), governs the propagation of weakly dispersive, weakly nonlinear water waves and serves as a model equation for any physical system for which the dispersion relation for frequency *vs.* wavenumber is approximated by $\omega/k = c_0(1-\beta k^2)$ and nonlinearity is weak and quadratic. It first appears explicitly in de Vries's dissertation (1894), although it is implicit in the work of Boussinesq (1872). Its current renaissance stems from the Fermi, Pasta & Ulam (1955) problem for a string of nonlinearly coupled oscillators, which, through the work of Zabusky & Kruskal and their colleagues, led to the discovery of the *soliton* and the development of inverse-scattering theory by Gardner *et al.* (1967). Many related evolution equations, each of which represents a balance between some form of dispersion (or variation of dispersion in the case of wave-packet evolution) and weak nonlinearity in an appropriate reference frame, have since been found to have properties analogous to those of the KdV equation – in particular, inverse-scattering solutions that are asymptotically dominated by solitons.

1. Introduction

The Korteweg–de Vries (KdV) equation

$$y_t + c_0\{y_x + \tfrac{3}{2}d^{-1}yy_x + \tfrac{1}{6}d^2 y_{xxx}\} = 0, \quad y = y(x,t), \tag{1.1}$$

where x is the horizontal distance from an arbitrary origin, t is the time, y is the free-surface displacement from the equilibrium level of an inviscid incompressible fluid of quiescent depth d, and $c_0 = (gd)^{\frac{1}{2}}$ is the speed of long, infinitesimal gravity waves, is usually attributed to Korteweg & de Vries (1895), although it appears, in somewhat different form, in the work of Boussinesq (1872). It may be reduced to the normal form

$$\eta_\tau + \eta\eta_\xi + \eta_{\xi\xi\xi} = 0, \quad \eta = \eta(\xi,\tau), \tag{1.2}$$

through the transformation

$$\xi = U^{\frac{1}{2}}\left\{\frac{x-(1+\alpha)c_0 t}{l}\right\}, \quad \tau = \tfrac{1}{6}U^{\frac{3}{2}}\left(\frac{d}{l}\right)^2\left(\frac{c_0 t}{l}\right), \quad \eta = \frac{3y-2\alpha d}{a}, \tag{1.3a,b,c}$$

where

$$U = 3al^2/d^3, \tag{1.4}$$

$a \ll d$ is an amplitude scale, $l \gg d$ is a horizontal length scale, and $\alpha = O(a/d)$ is an arbitrary constant. Nonlinearity and dispersion are measured by a/d and $(d/l)^2$, respectively, and U is a measure of their relative significance (Ursell 1953). The Boussinesq solitary wave [see (2.13) below], for which $U = O(1)$ as $a/d \downarrow 0$, represents a balance between nonlinearity, which tends to increase, and dispersion, which tends to decrease, the speed of a wave relative to the limiting value c_0.

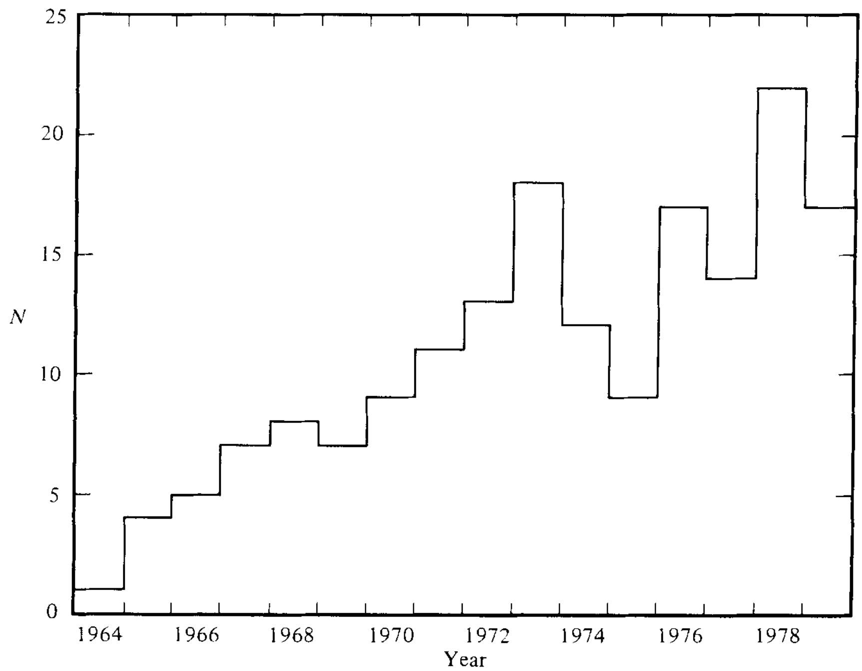

FIGURE 1. Citations (N) of Korteweg & de Vries (1895) by year (1964–1979).

The normalized KdV equation (1.2), has been described by Kruskal (1978) as 'arguably the simplest partial differential equation...not covered by classical methods.' It governs the evolution, in a reference frame moving with the basic wave speed $(1+\alpha)c_0$, of waves in any physical system for which c_0 is the speed of non-dispersive waves of infinitesimal amplitude, the dispersion relation for infinitesimal waves of wavenumber k is approximated by

$$\omega = c_0 k(1-\beta k^2) \quad (\beta \ll l^2) \tag{1.5}$$

($\beta = \frac{1}{6}d^2$ for gravity waves), and nonlinearity is weak and quadratic. The sign of $\eta_{\xi\xi\xi}$ must be changed if $\beta < 0$, as for capillary or certain plasma waves, and a balance between nonlinearity and dispersion then is possible only for $a < 0$. Closely related equations, which model other forms of nonlinearity and dispersion, are discussed in appendix A.

The first explicit appearance of the KdV equation is, as far as I have been able to determine, in de Vries's (1894) dissertation, which formed the basis of the much cited 1895 paper by Korteweg & de Vries. The principal citations of that paper prior to 1965 refer to cnoidal waves, however, and the KdV equation is not displayed explicitly in such standard references as Lamb (1932), Stoker (1957), and Wehausen & Laitone (1960). Current interest in the KdV equation stems from the facts that it can be solved exactly, for appropriately restricted initial data, by inverse-scattering theory (Gardner *et al.* 1967, 1974) and that the typical solution is asymptotically dominated by a set of *solitons* – i.e. solitary waves that retain their form after mutual interactions. The recent growth of this interest may be inferred from *Science Citation Index* counts of references to the 1895 paper, which are displayed in figure 1 (these counts presumably are only lower bounds to the total counts in the archival literature, but they do provide a reliable measure of the frequency of citation).

The present essay is devoted primarily to the pre-1965 history of the KdV equation with special reference to the often overlooked contributions of Boussinesq. I have not attempted a definitive coverage of the modern era, for which the reader may refer to reviews by Jeffrey & Kakutani (1972), Scott, Chu & McLaughlin (1973), Miura (1974, 1976), Kruskal (1974, 1975), Makhankov (1978), and Miles (1980), to Whitham's (1974, cha. 17) treatise, and to the seminal film by Zabusky, Kruskal & Deem (1965). Nor have I attempted to deal with the experimental confirmation of the predictions of the KdV equation, which go back to Russell (1845) and, in the modern era, include the work of Zabusky & Galvin (1971) and Hammack & Segur (1974).

2. Boussinesq's contributions

Joseph Valentin Boussinesq (1842–1929) received his doctorate from the Faculté des Sciences, Paris in 1867, occupied chairs at Lille from 1873 to 1885 and the Sorbonne from 1885 to 1896, and made significant contributions to the theories of hydrodynamics, elasticity, light, and heat (see Gillispie 1970; Rouse & Ince 1957). His name is often invoked in the literature of mechanics, but it appears that his papers are less often read – perhaps because his style seems ponderous to the modern ear and may have seemed so even to contemporary readers.†

Boussinesq's work on weakly nonlinear, weakly dispersive wave propagation is developed in three papers (1871*a*, *b*, 1872) and in his 1877 monograph. The first two papers are subsumed by the third (1872), and it is from that paper that the following account is abstracted [(2.1) and (2.3)–(2.5) below also are reported in the 1871*b* paper]; an extended quotation from Boussinesq's (1872) introduction is given in appendix B.

Boussinesq assumes irrotational flow in an incompressible, inviscid fluid, expands the velocity potential in powers of the vertical co-ordinate, and neglects higher-order terms to obtain the pair of equations

$$\int_x^\infty u_t dx = gy + \tfrac{1}{2}g(d^{-1}y^2 + d^2 y_{xx})_x \tag{2.1a}$$

and

$$y_t = -du_x - c_0(d^{-1}y^2 - \tfrac{1}{6}d^2 y_{xx})_x \tag{2.1b}$$

for the evolution of the free-surface displacement y and the horizontal velocity u at the bottom of the channel. [Boussinesq's notation is related to the present notation according to $h \equiv y$, $h_1 \equiv a$, $H \equiv d$, $u_0 \equiv u$, and $\omega \equiv c$. He, like his contemporaries,

† I am not alone in my appreciation of Boussinesq's contributions or of the difficulties of his style (see appendix B). Rouse & Ince (1957) conclude their biographical sketch with the statement that:

> Boussinesq's (1877) work not only was astonishingly complete for its time but it continues to be used as a basis for present-day analyses of a related nature. No one would read it from cover to cover for sheer enjoyment, for... much of the analysis now seems needlessly complex. But as a reference volume it remains both provocative and reasonably accurate, and the wealth of material which it embodies has by no means been exhausted.

uses the same symbols for ordinary and partial differentiation – e.g. $dh/dt \equiv \partial h/\partial t \equiv y_t$ here. $\int_x^\infty u_t dx \equiv \int_x^\infty u_t(x',t)\,dx'$.] Equation (2.1*b*) and the x derivative of (2.1*a*),

$$u_t = -gy_x - \tfrac{1}{2}g(d^{-1}y^2 + d^2 y_{xx})_{xx}, \tag{2.1c}$$

constitute one form of what are now known as Boussinesq's equations (*plural*) and describe both right- and left-running waves. Alternative forms may be obtained by introducing any of the depth-averaged velocity, the velocity potential at the bottom, or the velocity potential at the free surface as the dependent variable to be paired with y [the velocity potential at, and the displacement of, the free surface are canonical variables in Hamilton's sense (Broer 1974; Miles 1977)]. The neglect of the third and fourth terms, which represent nonlinearity and dispersion, respectively, in each of (2.1*b*) and (2.1*c*), yields the linearized, shallow-water equations, which imply the wave equation $y_{tt} = c_0^2 y_{xx}$.

The elimination of u between (2.1*a*) and (2.1*b*) yields

$$y_{tt} = c_0^2(y + \tfrac{1}{2}d^{-1}y^2 + \tfrac{1}{2}d^2 y_{xx})_{xx} - c_0(d^{-1}y^2 - \tfrac{1}{6}d^2 y_{xx})_{xt}, \tag{2.2}$$

which, like (2.1), describes both right-running and left-running waves. Boussinesq invokes the approximation $\partial_t \doteqdot -c_0\partial_x$ (y is a right-running wave) on the right-hand side of (2.2) to obtain

$$y_{tt} = c_0^2(y + \tfrac{3}{2}d^{-1}y^2 + \tfrac{1}{3}d^2 y_{xx})_{xx}, \tag{2.3}$$

which is now known as Boussinesq's equation (*singular*) and which he regards as the 'basis for all the following analysis' in his 1872 paper, even though it admits left-running waves that are inconsistent with the approximation $\partial_t \doteqdot -c_0\partial_x$ (as far as I have been able to determine, Boussinesq did not use these spurious solutions, but later workers occasionally have done so). In fact, the Boussinesq equations (2.1*b*) and (2.1*c*) or their equivalents have proved to be both more basic and more useful than (2.3).

Boussinesq's equation (2.3) may be reduced to the KdV equation (1.1) by factoring $c_0^2\partial_x^2 - \partial_t^2$ and approximating $c_0\partial_x - \partial_t$ by $2c_0\partial_x$, which is consistent with the reduction of (2.2) to (2.3), and then integrating with respect to x, but Boussinesq does not follow this path. Instead, he derives

$$c = c_0(1 + \tfrac{3}{4}d^{-1}y + \tfrac{1}{6}d^2 y^{-1} y_{xx}) \tag{2.4}$$

for the speed of a particular elevation of the wave, defined such that

$$y_t + (cy)_x = 0. \tag{2.5}$$

It is evident that (2.4), in which the terms $\tfrac{3}{4}d^{-1}y$ and $\tfrac{1}{6}d^2y^{-1}y_{xx}$ represent the contributions of weak nonlinearity and weak dispersion to c/c_0, is an implicit form of (1.1), which follows from the elimination of c between (2.4) and (2.5). But Boussinesq prefers to regard y as a function of

$$\sigma = \int_x^\infty y(x',t)\,dx' \tag{2.6}$$

and t, rather than x and t, and obtains the evolution equation

$$\frac{\partial y}{\partial t} = \frac{1}{4}\left(\frac{c_0}{d}\right)\frac{\partial}{\partial \sigma}\left\{y^3\left(1 + \tfrac{2}{3}d^3\frac{\partial^2 y}{\partial \sigma^2}\right)\right\}, \tag{2.7}$$

which is equivalent to (1.1); however, it does not appear to offer any advantages *vis-à-vis* (1.1) and is intrinsically more complicated in form. It seems, then, that Boussinesq obtained two implicit equivalents of the KdV equation, (2.4) and (2.7), but missed the simpler and more important (as it proved to be) form (1.1) of Korteweg & de Vries (1895).

The introduction of σ, although disadvantageous for the evolution equation, does offer advantages in the construction of integral invariants, and Boussinesq obtains

$$Q \equiv \sigma|_{x=x_0} = \int_{x_0}^{\infty} y\,dx, \tag{2.8}$$

$$E = \int_0^Q y\,d\sigma = \int_{x_0}^{\infty} y^2\,dx \tag{2.9}$$

and

$$M = \int_0^Q (yy_\sigma^2 - 3d^{-3}y^2)\,d\sigma = \int_{x_0}^{\infty} (y_x^2 - 3d^{-3}y^3)\,dx, \tag{2.10}$$

where x_0 (which may be replaced by $-\infty$ if appropriate) is defined such that $y = 0$ in $x < x_0$; the integrals may be replaced by averages over one wavelength for periodic disturbances. He also derives the equation of motion

$$\frac{d}{dt}\int_0^Q x\,d\sigma = \frac{d}{dt}\int_{x_0}^{\infty} xy\,dx \equiv Q\frac{dx_1}{dt} = c_0\left(Q + \frac{3}{4}\frac{E}{d}\right) \tag{2.11}$$

for the centre of mass x_1, which implies the fourth invariant (cf. Benjamin & Mahony 1971)

$$P = Qx_1 - \{Q + \tfrac{3}{4}(E/d)\}\,c_0 t. \tag{2.12}$$

The integral invariants Q and E represent mass and energy, respectively, the conservation of which follows from first principles. The invariants M, which Boussinesq calls the 'moment of instability', and P are less obvious, and their discovery by Boussinesq has been widely overlooked in the current literature (see, for example, Whitham 1965; Miura 1976).

Boussinesq then goes on to show that the hypothesis of uniform wave speed leads, through (2.4), to the solitary wave

$$y = a\,\mathrm{sech}^2\{(3a/d^3)^{\frac{1}{2}}(x-ct)\}, \quad c = c_0\{1 + \tfrac{1}{2}(a/d)\} \quad (a \ll d). \tag{2.13a, b}$$

The normalized form of (2.13*a*), as given by (1.3) with $\alpha = 0$ therein, is

$$\eta = 3\,\mathrm{sech}^2(\xi - \tau). \tag{2.14}$$

This hypothesis also yields a negative wave, with sech^2 replaced by $-\mathrm{cosech}^2$ in (2.13), but Boussinesq discards this singular solution as physically unacceptable [although it has proved to be significant for the construction of solutions through Bäcklund transformations (Wahlquist & Estabrook 1973)]. He then shows that the solitary wave may be derived from the conditional variational problem $\delta M = 0$ with E (and, implicitly, Q) fixed,† another discovery that has been widely overlooked.

Boussinesq closes his 1872 paper with a discussion, based on the description of nonlinearity and dispersion provided by (2.4) and on the requirement that M be a

† It is now known that $H = c_0 d^2 M/12$ is a Hamiltonian for the KdV equation (Gardner 1971; Zakharov & Faddeev 1972; Broer 1975; Flaschka & Newell 1975). See also Whitham (1974, §16.14) for a rather different variational approach to the KdV equation.

minimum for a wave of permanent form, of the evolution of a solitary wave, or a set of solitary waves, plus a dispersive tail, from an initial displacement of positive volume. He also shows that a solitary wave cannot evolve from an initial displacement of negative volume, for which nonlinearity and dispersion cannot achieve the balance that characterizes a solitary wave and, instead, lead to the formation of a dispersive wave train. His conclusions are in accord with the observations of Russell (1845) and Bazin (1865) and anticipate the predictions of inverse-scattering theory (see § 5).

3. The Korteweg & de Vries paper

Diederik Johannes Korteweg (1848–1941) was a student of J.D. van der Waals and received the first doctoral degree of the University of Amsterdam in 1878 for his dissertation on the motion of a viscous fluid in an elastic tube, with application to arterial blood flow. He occupied the chair of Mathematics and Mechanics at the University of Amsterdam from 1881 to 1918. His biographical memoir (Beth & Van der Woude 1946) does not mention his work on water waves, nor does it cite his 1895 paper with de Vries.

Korteweg appears to have believed that the paradox posed by the solitary wave, *vis-à-vis* the prediction of Airy's shallow-water theory that 'long waves in a rectangular canal must necessarily change their form as they advance, becoming steeper in front and less steep behind' (Korteweg & de Vries 1895), had not been adequately resolved by Boussinesq (1871*a*) and Rayleigh (1876), and it presumably was for this reason that he suggested the problem of long waves to his student Gustav de Vries. Neither Korteweg nor de Vries appears to have read Boussinesq's 1871*b* and 1872 papers.

Biographical data on Gustav de Vries are difficult to obtain (he is not to be confused with the Dutch mathematician H. de Vries), but it is known (van der Blij 1978) that he was a member of the Wiskundig Genootschap (Dutch Mathematical Society) from 1892, defended his thesis in 1894, subsequently taught at the Gymnasiums in Alkmaar and Haarlem, and published two papers on cyclones in the *Verhandlingen* of the Royal Dutch Academy of Arts and Sciences in 1896 and 1897. The 1895 paper of Korteweg & de Vries was excerpted and translated from de Vries's 1894 thesis.

Korteweg & de Vries, unlike Boussinesq, emphasize at the outset the advantages of working with unidirectional waves:

> First, then, we investigate the deformation of a system of waves of arbitrary shape but moving in one direction only, i.e. we consider one of the two systems of waves, starting in opposite directions in consequence of any disturbance, after their complete separation from each other. By adding to the motion of the fluid a uniform motion with velocity equal and opposite to the velocity of propagation of the waves, we may reduce the surface of such a system to approximate, but not perfect, rest.

This leads them, following the method used by Rayleigh ('whose paper has been of great influence on our researches') in his solution for the solitary wave,† to

$$Y_T = (g/D)^{\frac{1}{2}}(y_1 Y + \tfrac{3}{4}Y^2 + \tfrac{1}{6}\gamma D^3 Y_{XX})_X, \tag{3.1}$$

† Rayleigh's (1876) derivation of the equivalent of (2.13), which is reproduced by Lamb (1932, § 252), is rather more direct than that of Boussinesq (1872). Rayleigh notes in his *Scientific Papers* (vol. 1, p. 271) that he had been unaware, in 1876, of Boussinesq's (1871*a*) paper and that 'So far as our results are common, the credit of priority belongs of course to M. Boussinesq'.

where

$$T = t, \quad X = ct - x, \quad c = (gD)^{\frac{1}{2}}\{1 - D^{-1}(y_1 - y_0)\}, \tag{3.2a, b, c}$$

$$Y = y - y_0, \quad D = d + y_0, \tag{3.3a, b}$$

$$\gamma = 1 - 3(\rho g D^2)^{-1} T_1, \tag{3.4}$$

D is the minimum depth, y_0 is the minimum free-surface displacement (note that $y_0 = 0$, $Y = y$ and $D = d$ for a solitary wave; on the other hand, $y_0 < 0$ for a periodic wave), y_1 is a small (compared with D) but otherwise arbitrary length, and T_1 is the surface tension ($y \equiv \eta$, $D \equiv l$, $y_1 \equiv \alpha$, and $y \equiv 3\sigma/l^3$ in the notation of Korteweg & de Vries). It is consistent with the approximations implicit in (3.1) to approximate D by d except in $(gD)^{\frac{1}{2}} \doteqdot (gd)^{\frac{1}{2}}(1 + \frac{1}{2}d^{-1}y_0)$ in (3.2c) and to approximate Y by y except in the nonlinear term $Y^2 \doteqdot y^2 - 2y_0 y$; these changes, together with the neglect of surface tension, render (3.1) equivalent to (1.1) through the Galilean transformation (3.2). Korteweg & de Vries give special prominence to (3.1) by displaying it as the first equation in their introduction and by identifying it as 'this very important equation, to which we shall have frequently to revert in the course of this paper...'.

After deriving (3.1), Korteweg & de Vries obtain a solitary-wave solution that reduces to that of Boussinesq, (2.13) above, if $T_1 = 0$ and is negative if $D < (3T_1/\rho g)^{\frac{1}{2}}$ ($D < 0{\cdot}5$ cm for water). They also obtain the 'cnoidal wave' solution [Boussinesq (1877, p. 392) refers to the existence of such solutions but does not carry out the required integration]

$$Y = a\,\mathrm{cn}^2\{(3a/4\gamma m D^3)^{\frac{1}{2}} X \,|\, m\} \quad (\gamma > 0), \quad m = a/(a+b) \tag{3.5a, b}$$

and

$$c = (gD)^{\frac{1}{2}}\left[1 + \left(\frac{a}{mD}\right)\left\{\frac{1}{2} - \frac{E(m)}{K(m)}\right\}\right], \tag{3.5c}$$

where cn is a Jacobi elliptic cosine of squared modulus m in the notation of Abramowitz & Stegun (1964), $a \equiv -y_0$ ($y_0 < 0$), b is an arbitrary positive length ($a \equiv h$ and $b \equiv k$ in the notation of Korteweg & de Vries), E and K are complete elliptic integrals of the second and first kinds, and y_1 is implicitly defined by (3.2c) and (3.5c). The wave speed c is defined, following Stokes, such that the mean (over one wavelength) horizontal momentum of the flow is zero in the X, T reference frame. But Korteweg & de Vries, in comparing their cnoidal wave with Stokes's (1849) second-order approximation to a gravity wave, overlook the facts that Stokes uses the mean depth d and measures the free-surface displacement positive down, in consequence of which the third equation on p. 424 of their paper is *not*, despite their assertion, Stokes's result in their notation. It remains true, nevertheless, that the first three terms in the Fourier expansion of their cnoidal wave are equivalent to the first two terms in Stokes's expansion after correctly allowing for differences in notation and reference levels.

Korteweg & de Vries also obtain higher-order approximations to both solitary and cnoidal waves (cf. Laitone 1960); however, it is difficult to accept their implicit claim that these approximations provide a better resolution of the question of whether or not the Boussinesq solitary wave is one of permanent form than do the earlier results of Boussinesq (1871a) and Rayleigh (1876) or, especially, the qualitative arguments of Boussinesq (1872), of which they presumably were unaware. [A mathematical proof of the existence of the solitary wave was ultimately given by Friedrichs & Hyers (1954).]

In fine, the primary contributions of Korteweg & de Vries, *vis-à-vis* Boussinesq, were in working directly with unidirectional waves, the simpler form of their evolution equation – viz. *the* Korteweg–de Vries equation – and their direct solution of that equation for both solitary and periodic waves.

4. Inverse scattering theory

The renaissance of the KdV equation stems from the Fermi–Pasta–Ulam (FPU) problem for a string of nonlinearly coupled oscillators, the continuum limit of which is governed by a close counterpart of Boussinesq's equation (2.3). Fermi *et al.* (1955) considered a string of 64 masses coupled by weakly nonlinear springs as a model of a nonlinear, heat-conducting lattice. They had expected that an initial distribution of energy in the fundamental mode of this system would spread to the higher modes through the nonlinear coupling and hence lead ultimately to equipartition of energy among all the modes. Instead, their numerical solutions showed that, although energy was transferred to the first few of the lower modes, it was returned to the fundamental mode after 2000 cycles (fundamental periods). Recurrence (now called FPU recurrence) was not quite complete, but it was within the accuracy of the computation. Subsequent calculations by Tuck & Menzel (1972) and Abe & Abe (1979) have shown that recurrence is definitely incomplete and that there is a sequence of periods, $T_0 \ll T_1 \ll T_2 \ll \ldots$, where T_0 is the fundamental period, for approximate recurrence but that energy is ultimately transferred to more and more of the higher modes, such that equipartition is a plausible end state. Zakharov (1974) has shown that this quasi-recurrence phenomenon is associated with the near-integrability of the string of oscillators regarded as a Hamiltonian system.

The study of the FPU problem by Kruskal and Zabusky and their colleagues led ultimately to the discovery of the soliton (see Kruskal 1978) and to the exact solution of the KdV equation for suitably restricted initial data by the inverse-scattering algorithm of Gardner *et al.* (1967, 1974). Several recent reviews of this modern era are available (see last paragraph in § 1), and I consider it only briefly in the present account. The highlights are (in my view):

(i) The KdV equation admits an infinite number of integral invariants (Miura, Gardner & Kruskal 1968), beginning with those of Boussinesq (see § 2). This is closely connected with the interpretation of the KdV equation as a completely integrable Hamiltonian system (Gardner 1971; Zakharov & Faddeev 1972).

(ii) The solution of the KdV equation for an initial displacement that is sufficiently smooth and vanishes with sufficient rapidity as $x \to \pm\infty$† may be reduced to the solution of a linear integral equation (Gardner *et al.* 1967, 1974). Let

$$\eta(\xi, 0) = \eta_0(\xi) \quad (-\infty < \xi < \infty) \tag{4.1}$$

be the initial value of $\eta(\xi, \tau)$, the dimensionless variables being defined by (1.3); $b(k)$ be the reflexion coefficient determined by the solution of the direct scattering problem

$$\{(d/d\xi)^2 + k^2 + \tfrac{1}{6}\eta_0(\xi)\}\psi(\xi) = 0 \quad (-\infty < \xi < \infty), \tag{4.2}$$

$$\psi \sim e^{-ik\xi} + b(k)e^{ik\xi} \quad (\xi \uparrow \infty); \tag{4.3}$$

† This restriction rules out periodic waves. See Lax (1976) and references given there regarding solutions of the KdV equation that are periodic in x and 'almost periodic' in time, the simplest of which are cnoidal waves.

$\{\kappa_1, \kappa_2, \ldots, \kappa_N\}$ be the discrete spectrum of eigenvalues given by those poles of $b(k)$ that lie on the positive imaginary axis in the complex k plane (all poles of b must lie on the imaginary axis); c_n be the amplitude determined by

$$\psi_n \sim c_n \exp(-\kappa_n \xi) \quad (\xi \uparrow \infty), \tag{4.4}$$

where ψ_n is the eigenfunction for $k = i\kappa_n$, normalized according to

$$\int_{-\infty}^{\infty} \psi_n^2 \, d\xi = 1; \tag{4.5}$$

$$B(\xi;\tau) \equiv \sum_{n=1}^{N} c_n^2 \exp(8\kappa_n^2 \tau - \kappa_n \xi) + (2\pi)^{-1} \int_{-\infty}^{\infty} b(k) \exp\{i(8k^3\tau + k\xi)\} dk; \tag{4.6}$$

$K(x, y; \tau)$ be determined by the Marchenko integral equation

$$K(x, y; \tau) + B(x+y; \tau) + \int_x^{\infty} B(y+z; \tau) K(x, z; \tau) dz = 0. \tag{4.7}$$

Then the solution of the initial-value problem posed by (1.2) and (4.1) is given by

$$\eta(\xi, \tau) = 12(d/d\xi) K(\xi, \xi; \tau). \tag{4.8}$$

(iii) $N \geqslant 1$ if the initial volumetric displacement is non-negative ($Q \geqslant 0$), and the asymptotic solution then is dominated by the N solitary waves (or *solitons* – see below) associated with the discrete spectrum:

$$\eta \sim 12 \sum_{n=1}^{N} \kappa_n^2 \operatorname{sech}^2\{\kappa_n \xi - 4\kappa_n^3 \tau + \delta_n\} \quad (\tau \uparrow \infty), \tag{4.9}$$

where the constants $\delta_1, \delta_2, \ldots, \delta_N$ depend implicitly on both the discrete and continuous spectra. The asymptotic solution also comprises a dispersive wave train, which is associated with the continuous spectrum. The amplitude of this wave train is small compared with the amplitudes, but the mass and energy may be comparable with those, of the solitary waves.

(iv) These solitary waves – or any set of unidirectional, Boussinesq solitary waves of different amplitudes, and hence of different speeds – pass through one another without any permanent change of shape and suffer only phase shifts, even though nonlinear distortion is quite significant during the mutual interaction(s). This phenomenon was discovered from numerical solutions of the KdV equation by Zabusky & Kruskal (1965), who named the corresponding solitary waves *solitons*; it is beautifully displayed in the film by Zabusky *et al.* (1965) and was analytically confirmed by Lax (1968) and Gardner *et al.* (1967).

These are capital discoveries, which have since been extended to a wide class of evolution equations [see Lax (1968), Zakharov & Shabat (1972), Ablowitz *et al.* (1973) and appendix A below]. It must be emphasized, nevertheless, that exact analytical solutions of the Marchenko integral equation appear to be possible only for initial data for which the dispersive wave train is absent and that these solutions can be obtained more directly by Hirota's (1971, 1976) algorithm; moreover, rather efficient programs for the direct, numerical solution of the KdV equation now exist [see Fornberg & Whitham (1978) and the references given by Zabusky (1980)]. It may be that more general solutions of the Marchenko integral equation await discovery, and analytical solutions certainly provide insights that are not attainable through numerical solutions, but it also may be that the major benefits of inverse-scattering theory for fluid mechanics have now been realized.

This work was partially supported by the Physical Oceanography Division, National Science Foundation (NSF Grant OCE77-24005) and by a contract with the Office of Naval Research. I am indebted to L. J. F. Broer, J. Korevaar and G. Ooms for biographical data on Korteweg and de Vries, to W.K. Melville, R.M. Miura, D.H. Peregrine, and N.J. Zabusky for helpful comments, and to A.C. Vastano for permission to reproduce the translation in appendix B.

Appendix A. Related equations

Modified KdV equations

The nonlinear term $\eta\eta_\xi$ in (1.2) must be replaced by $\eta^p\eta_\xi$ if the nonlinearity is of degree $p+1$. The most important case, other than $p = 1$, is $p = 2$, and the resulting equation (after replacing η by ζ),

$$\zeta_\tau + \zeta^2\zeta_\xi + \zeta_{\xi\xi\xi} = 0, \tag{A 1}$$

is known as the modified Korteweg-de Vries (mKdV) equation. Moreover, the sign of the nonlinear term may be changed to obtain the non-trivial alternative

$$\zeta_\tau - \zeta^2\zeta_\xi + \zeta_{\xi\xi\xi} = 0. \tag{A 2}$$

[Note that changing the sign of the nonlinear term in (1.2) yields nothing new, since the resulting equation is reduced to (1.2) by changing the sign of η.] Equation (A 1) admits the solitary-wave solution [cf. (2.14)]

$$\zeta = \sqrt{6}\,\mu\,\mathrm{sech}\,(\mu\xi - \mu^3\tau), \tag{A 3}$$

where μ is a family parameter. This solution represents a balance between weak dispersion, as described by (1.5), and cubic nonlinearity, a balance that is impossible for (A 2), which has no solitary-wave solutions.

Miura (1968) discovered that the solutions of (A 2) are mapped into solutions of (1.2) through the transformation

$$\eta = -\zeta^2 - \sqrt{6}\,\zeta_\xi. \tag{A 4}$$

It should be emphasized, however, that the entire set of those solutions of (A 2) that vanish with sufficient rapidity as $\xi \to \pm\infty$ map into a sparse subset of all of those solutions of (1.2) that vanish as $\xi \to \pm\infty$ (Ablowitz & Kruskal 1979). Miura's transformation (A 4) played a key role in the development of inverse-scattering theory.

Inverse-scattering theory also may be invoked to solve both the mKdV equation (Wadati 1972, 1973) and the generalized KdV equation obtained by adding the term $-C\eta^2\eta_\xi$ to (1.2), where C is a constant that measures cubic nonlinearity, which may be significant if quadratic linearity is sufficiently weak, as in some internal wave problems (Miles 1979). It is not applicable to the equation obtained by replacing $\eta\eta_\xi$ by $\eta^p\eta_\xi$ with $p > 2$ (Miura 1976).

BBM equation

The dispersion term $c_0 y_{xxx}$ in the KdV equation (1.1) may be replaced by $-y_{xxt}$ without altering the order of the approximation. The result,

$$y_t + c_0(1 + \tfrac{3}{2}d^{-1}y)\,y_x - \tfrac{1}{6}d^2 y_{xxt} = 0, \tag{A 5}$$

appears to be due originally to Peregrine (1966), although it is commonly known either as the *BBM equation* by virtue of an extensive study by Benjamin, Bona & Mahony (1972) or as the *regularized long-wave equation*, the designation proposed by Benjamin *et al.* Its linear counterpart is associated with the dispersion relation

$$\omega = c_0 k(1 + \tfrac{1}{6}k^2 d^2)^{-1}, \tag{A 6}$$

which, in contrast to (1.5), is positive for all k [although consistent with the derivation of (A 5) only for $k^2 d^2 \gg 1$, in which domain it is equivalent to (1.5)]. This may be advantageous for numerical work; on the other hand, (A 5) does not admit an infinite number of integral invariants and cannot be solved by inverse scattering theory.† Broer (1975, 1976) has discussed (A 5) and generalizations thereof from the viewpoint of Hamiltonian theory. See also Kruskal (1975).

Perturbed KdV equations

The evolution of a weakly dispersive, weakly nonlinear wave in a channel of slowly varying breadth and depth, for which the scale of the slow variation must be large compared with the length scale l of the wave, is governed by an equation of the form (1.1) with variable coefficients and an additional term that is proportional to y (Shuto 1974; Ostrovsky & Pelinovsky 1975), although x and t then are no longer simple space and time co-ordinates. Damping may be similarly incorporated (Miles 1976). If the resulting perturbation of the KdV equation is small, the solution may be obtained by a perturbation of inverse-scattering theory (Karpman & Maslov 1977; Kaup & Newell 1978; Karpman 1979); the details are complicated, and explicit results have been obtained only for perturbed solitary waves.

Benjamin–Ono equation

Benjamin (1967) and Whitham (1967; 1974, §13.14) independently suggested that weakly nonlinear waves for which the wave speed is $c(k)$ in the linear regime are governed by

$$y_t + \kappa c_0 y y_x + \frac{1}{2\pi}\int_{-\infty}^{\infty} c(k)\,dk \int_{-\infty}^{\infty} e^{ik(x-\xi)}\,y_\xi(\xi,t)\,d\xi, \tag{A 7}$$

where κ is an inverse length (if y is a displacement). Equation (A 7) provides an exact description of dispersion if $\kappa = 0$. It reduces to the KdV equation (1.1) for weakly nonlinear, shallow-water gravity waves if

$$\kappa = \tfrac{3}{2}d^{-1}, \quad c = c_0(1 - \tfrac{1}{6}k^2 d^2). \tag{A 8}$$

† It should be noted that (A 5) reduces to (1.2) under the transformation (1.3) if and only if $O(d^2/l^2)$ is neglected. Such neglect is, of course, consistent with the derivation of (A 5) for long waves but eliminates the putative advantage of (A 5) for short waves.

Benjamin (1967) considered internal waves in a thin, stratified layer embedded in an otherwise homogeneous fluid of (in the simplest case) unlimited extent, for which

$$c(k) = c_0(1-\gamma|k|) \tag{A 9}$$

and γ depends on the density distribution. The substitution of (A 9) into (A 7) yields the Benjamin–Ono† equation

$$y_t + c_0(1+\kappa y)y_x = \frac{\gamma c_0}{\pi}\int_{-\infty}^{\infty}\frac{y_\xi\, d\xi}{x-\xi}, \tag{A 10}$$

wherein the integral is a Cauchy principal value. Benjamin showed that (A 10) admits the solitary-wave solution

$$y = a\lambda^2\{(x-ct)^2+\lambda^2\}^{-1}, \tag{A 11}$$

wherein

$$c = c_0(1+\tfrac{1}{4}\kappa a), \quad \lambda = 4\gamma(\kappa a)^{-1}, \tag{A 12a,b}$$

a is the amplitude, and y is the streamline displacement at some reference level.

Joseph (1977) has generalized (A 10) to obtain the evolution equation for internal waves in a stratified layer in a fluid of finite depth D and has obtained the corresponding solitary-wave solution, which reduces to (A 11) in the limit $D \uparrow \infty$ and to the equivalent of (2.13) in the limit $D \downarrow 0$; see also Henyey (1980). Satsuma, Ablowitz & Kodama (1979) and Chen, Hirota & Lee (1980) have shown that Joseph's equation, and hence also the Benjamin–Ono equation, admits an infinite number of integral invariants and an inverse-scattering-theory algorithm.

Nonlinear Schrödinger equation

The assumption of the wave packet

$$\zeta = \mathrm{Re}\{A(x,t)\exp[i(\omega_0 t - k_0 x)]\}, \tag{A 13}$$

where $A(x,t)$ is a slowly varying, complex amplitude and $\omega_0 \equiv \Omega_0(k_0)$ and k_0 are the carrier frequency and wavenumber, and the nonlinear dispersion relation

$$\omega = \Omega_0(k) + a^2\Omega_2(k) \tag{A 14}$$

leads to the nonlinear Schrödinger equation (Whitham 1974, § 17.8)

$$A_t + \Omega_0' A_x + \tfrac{1}{2}i\Omega_0'' A_{xx} - i\Omega_2|A|^2 A = 0, \tag{A 15}$$

where $\Omega_0' \equiv d\Omega_0/dk$ is the group velocity. (Note that, in this case, the evolution in a reference frame moving with the group velocity represents a balance between weak variations of the dispersion.) This equation admits solitary-wave solutions (which represent *envelope solitons*) and may be solved by inverse-scattering theory (Zakharov & Shabat 1972).

Appendix B. Boussinesq's (1872) introduction

The following quotation from Boussinesq's introduction to his 1872 paper is taken from the translation by Vastano & Mungall (1976), who remark that '[our] translation attempts to preserve the sense of the nineteenth century French language without the

† Equation (A 10) is equivalent to Benjamin's (1.11) after invoking his (A 4) but does not appear explicitly in his 1967 paper. It was subsequently derived and explicitly displayed by Ono (1975).

introduction of modern terminology, and we have thus retained Boussinesq's somewhat heavy phraseology'.

I propose to give here an almost complete theory concerning [solitary waves] taking for the point of departure of my analysis the characteristic which essentially distinguishes them from other undulatory movements of fluids. This characteristic consists of the fact that the horizontal velocities of the fluid parcels are approximately equal over the entire extent of the same normal cross section of the canal. [This] permits one to obtain [the velocity potential] ϕ in a convergent series containing no other unknowns than the velocity at the different points of the bottom. This series, substituted for ϕ in the known formulas of hydrodynamics, furnishes, using the depth from the bottom of the different normal cross sections and the velocity at the different points of the bottom, two equations with partial derivatives with respect to time and with respect to the longitudinal co-ordinate x [(2.1 a, b) above]. As a first approximation, that is to say neglecting all terms that are very small in comparison with the ratio of the intumescence† height to the original depth, these two equations lead to the remarkable law of Lagrange...that all intumescenses of small height, positive or negative, propagate conserving their form and with a velocity equal to the square root of the product of the gravity by the original depth.

This law ceases to be true in a second approximation. Then, except for certain particular phenomena, a velocity of propagation common to the whole wave no longer holds, and it is expedient to divide the intumescence, beginning at its front, into infinitely small parts of constant volume, included between consecutive planes normal to the axis of the canal, and positive or negative according to their height, which is the excess of the instantaneous depth at the points considered over the original depth. Separately considering each of these parts, I will show that the square of its velocity of propagation is equal, at any given instant, to the product of the gravity g by the sum: (1) of the original depth, (2) of one and a half times the instantaneous height of the part of the intumescence considered, and (3) of the curvature assumed by the free surface, multiplied by the inverse of the same height and by one-third the cube of the original depth [(2.4) above].

It is particularly interesting to study the movement of the general center of gravity of the wave.... [T]he height, positive or negative, of the center of gravity considered above the initial free surface...stays constant throughout the motion: It is invariable in that when multiplied by twice the volume of the intumescence and by the weight of a unit volume of the liquid, it becomes approximately equal to the total and constant energy of the wave....

The total volume of an intumescence and its energy [(2.8) and (2.9) above] are not the only two integrals which remain constant during the motion, and which thus characterize each intumescence. There is in addition a third, the elements of which one obtains by dividing, as previously stated, the intumescence into infinitely small parts by normal cross sections, and by multiplying the distance between two consecutive sections by the square of the slope that the instantaneous free surface makes with the original free surface, then subtracting three times the cube of the ratio of the height of the intumescence to the initial depth [(2.10) above]. I call this integral the *moment of instability of the wave*....

† Boussinesq's term for a moving disturbance of the free surface.

It is natural to inquire, amongst all the possible forms that can be assumed by... an intumescence, if one exists which gives to all parts the same velocity of propagation, in such a way that the wave advances without deformation. If there is such a form, it is probable that waves will frequently acquire it in the initial period of motion.... I will show now that such a form indeed exists when the intumescence is positive and of sufficiently small volume [and that it is], with all the circumstances which characterize it, the solitary wave of Scott Russell.

It remains to explain the frequent formation of this particular species of wave and their unique stability. I succeed in demonstrating that a solitary wave is, among all the intumescences of equal energy, that for which the integral [M] that I have called the *moment of instability* has its smallest value, and is also the only one which makes this integral maximum or minimum. The result is that, if an intumescence differs little at a given instant, with respect to the shape, from a solitary wave of the same energy, it will differ little at all times: for it cannot deviate notably without its moment of instability growing in an appreciable manner, which is impossible, since this moment is invariable. The true form of the intumescence will thus oscillate continuously about that of a solitary wave, or rather the friction that we have neglected, and which has a notable influence at the beginning of the propagation, will not delay in cancelling its small variations and in giving to the wave a permanent form.... And, just as friction can restore certain bodies after large excursions to their state of equilibrium, one can imagine that it may also be able to change intumescences of considerably differing shape into solitary waves.

Among several intumescences of like energy, those whose profiles deviate the most from that of a solitary wave, and which as a result differ most in appearance from one moment to the next, have, as has been shown, the largest moments of instability: these moments therefore well deserve the name that I have given them, since they indicate in some manner, by their relative magnitude, the speed and amplitude of the transformations that each wave undergoes.

The production of a stable form is impossible in the two cases of a negative wave and a continuous intumescence, and also in the case of a positive intumescence of too short a nature, which could not change into a unique solitary wave without acquiring an excessive height and consequently becoming unstable. In these three cases, the theory permits one, not only to obtain experimental formulas for the speeds of propagation, but moreover to explain the different circumstances which produce them....

REFERENCES

ABE, K. & ABE, T. 1979 Recurrence of initial state of the Korteweg-de Vries equation. *Phys. Fluids* **22**, 1644–1646.

ABLOWITZ, M. J., KAUP, D. J., NEWELL, A. C. & SEGUR, H. 1974 The inverse scattering transform – Fourier analysis for nonlinear problems. *Stud. Appl. Math.* **53**, 249–315.

ABLOWITZ, M. J. & KRUSKAL, M. 1979 A note on Miura's transformation. *J. Math. Phys.* **20**, 999–1003.

ABRAMOWITZ, M. & STEGUN, I. 1964 *Handbook of Mathematical Functions*. Washington: National Bureau of Standards.

BAZIN, H. 1865 Recherches expérimentales sur la propagation des ondes. *Mém. présentés par divers Savants à L'Acad. Sci. Inst. France* **19**, 495–644.

BENJAMIN, T. B. 1967 Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.* **29**, 559–592.

BENJAMIN, T. B., BONA, J. L. & MAHONY, J. J. 1972 Model equations for long waves in nonlinear dispersive systems. *Phil. Trans. Roy. Soc.* A **272**, 47–78.

BENJAMIN, T. B. & MAHONY, J. J. 1971 On an invariant property of water waves. *J. Fluid Mech.* **49**, 385–389.

BETH, H. & VAN DER WOUDE, W. 1946 Levensbericht van D. J. Korteweg (31 Maart 1848–10 Mai 1941). *Jaarboek Kon. Ned. Akad. v. Wetenschappen*, 1945–46. Amsterdam.

BOUSSINESQ, J. 1871*a* Théorie de l'intumescence liquide, appelée onde solitaire ou de translation, se propageant dans un canal rectangulaire. *C. r. Acad. Sci. Paris* **72**, 755–759.

BOUSSINESQ, J. 1871*b* Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal. *C. r. Acad. Sci. Paris* **73**, 256–260.

BOUSSINESQ, J. 1872 Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *J. Math. Pures Appl.* **17** (2), 55–108; see also Vastano & Mungall (1976).

BOUSSINESQ, J. 1877 Essai sur la théorie des eaux courantes. *Mém. présentés par divers Savants à L'Acad. Sci. Inst. France* (*séries* 2) **23**, 1–680 (see also **24**, 1–64).

BROER, L. J. F. 1974 On the Hamiltonian theory of surface waves. *Appl. Sci. Res.* **29**, 430–446.

BROER, L. J. F. 1975*a* Hidden Hamiltonians of first-order equations. *Physica* A **79**, 583–596.

BROER, L. J. F. 1975*b* Approximate equations for long water waves. *Appl. Sci. Res.* **31**, 377–395.

BROER, L. J. F. 1976 Stable model equations for long water waves. *Appl. Sci. Res.* **32**, 619–636.

CHEN, H. H., HIROTA, R. & LEE, Y. C. 1980 Inverse scattering problem for internal waves with finite fluid depth. *Phys. Lett.* A **75**, 254–256.

DE VRIES, G. 1894 Bijdrage tot de kennis der lange golven. Doctoral dissertation, University of Amsterdam.

FERMI, E., PASTA, J. & ULAM, S. 1955/1974 Studies of nonlinear problems. I. *Nonlinear Wave Motion. Lectures in Applied Mathematics*, vol. 15 (ed. A. C. Newell), pp. 143–156. Amer. Math. Soc., Providence, R.I.

FLASCHKA, H. & NEWELL, A. 1975 Integrable systems of nonlinear evolution equations. *Dynamical Systems, Theory and Applications* (ed. J. Moser), pp. 355–440. Springer.

FORNBERG, B. & WHITHAM, G. B. 1978 A numerical and theoretical study of certain nonlinear wave phenomena. *Phil. Trans. Roy. Soc.* A **289**, 373–404.

FRIEDRICHS, K. O. & HYERS, D. H. 1954 The existence of solitary waves. *Comm. Pure Appl. Math.* **7**, 517–550.

GARDNER, C. S. 1971 Korteweg–de Vries equation and generalizations. IV. The Korteweg–de Vries equation as a Hamiltonian system. *J. Math. Phys.* **12**, 1548–1551.

GARDNER, C. S., GREENE, J. M., KRUSKAL, M. D. & MIURA, R. M. 1967 Method for solving the Korteweg–de Vries equation. *Phys. Rev. Lett.* **19**, 1095–97.

GARDNER, C. S., GREENE, J. M., KRUSKAL, M. D., & MIURA, R. M. 1974 Korteweg–de Vries equation and generalizations. VI. Methods for exact solution. *Comm. Pure Appl. Math.* **27**, 97–133.

GILLISPIE, C. C. 1970 *Dictionary of Scientific Biography*. Scribners.

HAMMACK, J. L. & SEGUR, H. 1974 The Korteweg–de Vries equation and water waves. Part 2. Comparison with experiments. *J. Fluid Mech.* **65**, 289–314.

HENYEY, F. S. 1980 Finite-depth and infinite-depth internal-wave solitons. *Phys. Rev.* A **21**, 1054–1056.

HIROTA, R. 1971 Exact solution of the Korteweg–de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.* **27**, 1192–1194.

HIROTA, R. 1976 Direct method of finding exact solutions of nonlinear evolution equations. *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications* (ed. R. M. Miura), pp. 40–68. Springer.

JEFFREY, A. & KAKUTANI, T. 1972 Weak nonlinear dispersive waves: a discussion centered around the Korteweg–de Vries equation. *SIAM Rev.* **14**, 582–643.

JOSEPH, R. I. 1977 Solitary waves in a finite-depth fluid. *J. Phys.* A **10**, L225–227.

KARPMAN, V. I. 1979 Soliton evolution in the presence of perturbation. *Physica Scripta* **20**, 462–478.

KARPMAN, V. I. & MASLOV, E. M. 1977 A perturbation theory for the Korteweg-de Vries equation. *Phys. Lett.* A **60**, 307–308.

KAUP, D. J. & NEWELL, A. C. 1978 Solitons as particles, oscillators, and in slowly changing media: a singular perturbation theory. *Proc. Roy. Soc.* A **361**, 413–446.

KORTEWEG, D. J. & DE VRIES, G. 1895 On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.* **39**, 422–443.

KRUSKAL, M. D. 1974 The Korteweg-de Vries equation and related evolution equations. *Nonlinear Wave Motion. Lectures in Applied Mathematics*, vol. 15 (ed. A. C. Newell), pp. 61–84. Am. Math. Soc., Providence, R.I.

KRUSKAL, M. D. 1975 Nonlinear wave equations. *Dynamical Systems Theory and Applications* (ed. J. Moser), pp. 310–354. Springer.

KRUSKAL, M. D. 1978 The birth of the soliton. *Nonlinear Evolution Equations Solvable by the Spectral Transform* (ed. Francesco Calogero), pp. 1–8. Pitman.

LAITONE, E. V. 1960 The second approximation to cnoidal and solitary waves. *J. Fluid Mech.* **9**, 430–444.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LAX, P. D. 1968 Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.* **21**, 467–490.

LAX, P. D. 1976 Almost periodic solutions of the KdV equation. *SIAM Rev.* **18**, 351–375.

MAKHANKOV, V. G. 1978 Dynamics of classical solitons (in non-integrable systems). *Phys. Rep.* **35**, 1–128.

MILES, J. W. 1976 Korteweg-de Vries equation modified by viscosity. *Phys. Fluids* **19**, 1063.

MILES, J. W. 1977 On Hamilton's principle for surface waves. *J. Fluid Mech.* **83**, 153–158.

MILES, J. W. 1979 On internal solitary waves. *Tellus* **31**, 456–462.

MILES, J. W. 1980 Solitary waves. *Ann. Rev. Fluid Mech.* **12**, 11–43.

MIURA, R. M. 1968 Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Math. Phys.* **9**, 1202–1204.

MIURA, R. M. 1974 The Korteweg-de Vries equation: a model equation for nonlinear dispersive waves. *Nonlinear Waves* (eds. S. Leibovich & A. R. Seebass), pp. 212–234. Cornell University Press.

MIURA, R. M. 1976 The Korteweg-de Vries equation: a survey of results. *SIAM Rev.* **18**, 412–459.

MIURA, R. M., GARDNER, C. S., & KRUSKAL, M. D. 1968 Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Math. Phys.* **9**, 1204–1209.

ONO, H. 1975 Algebraic solitary waves in stratified fluids. *J. Phys. Soc. Japan* **39**, 1082–1091.

OSTROVSKY, L. A. & PELINOVSKY, E. N. 1975 Refraction of nonlinear ocean waves in a beach zone. *Atmos. & Oceanic Phys.* **11**, 37–41.

PEREGRINE, D. H. 1966 Calculations of the development of an undular bore. *J. Fluid Mech.* **25**, 321–330.

RAYLEIGH, LORD 1876 On waves. *Phil. Mag.* **1**, 257–279; Scientific Papers, vol. 1, pp. 251–271.

ROUSE, H. & INCE, S. 1957 *History of Hydraulics*. Dover.

RUSSELL, J. S. 1845 Report on waves. *Rep. Meet. Brit. Assoc. Adv. Sci., 14th, York*, 1844, pp. 311–390. London: John Murray.

SATSUMA, J., ABLOWITZ, M. & KODAMA, Y. 1979 On an internal wave equation describing a stratified fluid with finite depth. *Phys. Lett.* A **73**, 283–286.

SCOTT, A. C., CHU, F. Y. F. & MCLAUGHLIN, D. W. 1973 The soliton: a new concept in applied science. *Proc. I.E.E.E.* **61**, 1443–1483.

SHUTO, N. 1974 Non-linear long waves in a channel of variable section. *Coastal Eng. Japan* **17**, 1–12.

STOKER, J. J. 1957 *Water Waves*. Interscience.

TUCK, J. L. & MENZEL, M. T. 1972 The superperiod of the nonlinear weighted string (FPU) problem. *Advances in Maths* **9**, 399–407.

URSELL, F. 1953 The long-wave paradox in the theory of gravity waves. *Proc. Camb. Phil. Soc.* **49**, 685–694.

VAN DER BLIJ, F. 1978 Some details of the history of the Korteweg–de Vries equation. *Nieuw Archief Voor Wiskunde* **26**, 54–64.

VASTANO, A. C. J. & MUNGALL, J. C. H. 1976 Theory of waves and surges which propagate the length of a horizontal rectangular canal, imparting to the fluid contained within the canal velocities that are sensibly the same from the top to the bottom [English translation of Boussinesq (1872)]. *College of Geosciences, Texas A&M Univ. Rep.* 76-2-T.

WADATI, M. 1972 The exact solution of the modified Korteweg–de Vries equation. *J. Phys. Soc. Japan* **32**, 1681.

WADATI, M. 1973 The modified Korteweg–de Vries equation. *J. Phys. Soc. Japan* **34**, 1289–1296.

WAHLQUIST, H. D. & ESTABROOK, F. B. 1973 Bäcklund transformations for solutions of the Korteweg–de Vries equation. *Phys. Rev. Lett.* **31**, 1386–1390.

WEHAUSEN, J. V. & LAITONE, E. V. 1960 Surface waves. *Encyc. Phys.* **9**, 446–778.

WHITHAM, G. B. 1965 Non-linear dispersive waves. *Proc. Roy. Soc.* A **283**, 238–261.

WHITHAM, G. B. 1967 Variational methods and applications to water waves. *Proc. Roy. Soc.* A **299**, 6–25.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley–Interscience.

ZABUSKY, N. J. & GALVIN, C. J. 1971 Shallow-water waves, the Korteweg–de Vries equation and solitons. *J. Fluid Mech.* **47**, 811–824.

ZABUSKY, N. J. & KRUSKAL, M. D. 1965 Interaction of 'solitons' in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* **15**, 240–243.

ZABUSKY, N. J., KRUSKAL, M. D. & DEEM, G. S. 1965 *Formation, Propagation and Interaction of Solitons (Numerical Solutions of Differential Equations Describing Wave Motion in Nonlinear Dispersive Media).* A 16 mm, 35 min silent, black-and-white film available on loan from the Bell Telephone Laboratories, Inc. Film Library, Murray Hill, N.J. 07971.

ZAKHAROV, V. E. 1974 On stochastization of one-dimensional chains of nonlinear oscillators. *Sov. Phys., J. Exp. Theor. Phys.* **38**, 108–110.

ZAKHAROV, V. E. & FADDEEV, L. D. 1972 Korteweg–de Vries equation: a completely integrable Hamiltonian system. *Functional Anal. Appl.* **5**, 280–287.

ZAKHAROV, V. E. & SHABAT, A. B. 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys., J. Exp. Theor. Phys.* **34**, 62–69.